

Coherent States and Duality

José M. Isidro

Department of Theoretical Physics,
1 Keble Road, Oxford OX1 3NP, UK

and

Instituto de Física Corpuscular (CSIC-UVEG)
Apartado de Correos 22085, 46071 Valencia, Spain
isidro@thphys.ox.ac.uk

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Abstract

We formulate a relation between quantum–mechanical coherent states and complex–differentiable structures on the classical phase space \mathcal{C} of a finite number of degrees of freedom. Locally–defined coherent states parametrised by the points of \mathcal{C} exist when there is an almost complex structure on \mathcal{C} . When \mathcal{C} admits a complex structure, such coherent states are globally defined on \mathcal{C} . The picture of quantum mechanics that emerges allows to implement duality transformations.

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1 Introduction

Duality in strings, branes and M–theory [1, 2] motivates the study of duality in quantum mechanics. Now scattering amplitudes in perturbative string theory can be organised in a manner analogous to the loop expansion of quantum field theory. The latter expansion is a series in powers of \hbar , with an L –loop amplitude contributing a prefactor \hbar^{L-1} . In string theory, the role of loops is played by the genus of the string worldsheet, and the role of \hbar is played by the string coupling g_s . By the same token, duality in quantum mechanics refers to different regimes in a series expansion in powers of \hbar .

In the standard formulation of quantum mechanics, if one observer calls a certain phenomenon *semiclassical*, then so will it be for all other observers. If one observer calls a certain phenomenon *strong quantum*, then so will it be for all other observers. This does not allow for the relativity of the concept of a quantum that underlies the notion of duality. In view of these developments, a framework for quantum mechanics is required that can accommodate such a relativity of the concept of a quantum. In this letter we address this issue through an analysis of coherent states [3].

2 Global coherent states

Throughout this letter, \mathcal{C} will denote a $2n$ -dimensional classical phase space endowed with a symplectic form $\omega_{\mathcal{C}}$ that, in local Darboux coordinates, may be written as

$$\omega_{\mathcal{C}} = \sum_{l=1}^n dp_l \wedge dq^l. \quad (1)$$

Let us assume that \mathcal{C} admits a complex structure $\mathcal{J}_{\mathcal{C}}$. Furthermore let $\mathcal{J}_{\mathcal{C}}$ be compatible with the symplectic structure $\omega_{\mathcal{C}}$. This means that the real and imaginary parts of the holomorphic coordinates z^l for $\mathcal{J}_{\mathcal{C}}$ are Darboux coordinates for the symplectic form $\omega_{\mathcal{C}}$:

$$z^l = q^l + ip_l, \quad l = 1, \dots, n. \quad (2)$$

The set of all z^l so defined provides a holomorphic atlas for \mathcal{C} . Upon quantisation, the Darboux coordinates q^l and p_l become operators Q^l and P_l satisfying the Heisenberg algebra

$$[Q^j, P_k] = i\delta_k^j. \quad (3)$$

Define the annihilation operators

$$A^l = Q^l + iP_l, \quad l = 1, \dots, n. \quad (4)$$

Quantum excitations are measured with respect to a vacuum state $|0\rangle$. The latter is defined as that state in the Hilbert space \mathcal{H} which satisfies

$$A^l|0\rangle = 0, \quad l = 1, \dots, n, \quad (5)$$

and coherent states $|z^l\rangle$ are eigenvectors of A^l , with eigenvalues given in equation (2) above:

$$A^l|z^l\rangle = z^l|z^l\rangle, \quad l = 1, \dots, n. \quad (6)$$

How do the vacuum state $|0\rangle$ and the coherent states $|z^l\rangle$ transform under a canonical coordinate transformation on \mathcal{C} ? Call the new Darboux coordinates q'^l, p'_l . Upon quantisation the corresponding operators Q'^l, P'_l continue to satisfy the Heisenberg algebra (3). Then the combinations

$$z'^l = q'^l + ip'_l, \quad l = 1, \dots, n \quad (7)$$

continue to provide holomorphic coordinates for \mathcal{C} , and the transformation between the z^l and the z'^l is given by an n -variable holomorphic function f ,

$$z' = f(z), \quad \bar{\partial}f = 0. \quad (8)$$

We can write as above

$$A'^l = Q'^l + iP'_l, \quad l = 1, \dots, n, \quad (9)$$

$$A'^l|0\rangle = 0, \quad l = 1, \dots, n, \quad (10)$$

$$A^l|z^l\rangle = z^l|z^l\rangle, \quad l = 1, \dots, n. \quad (11)$$

There is no physical difference between equations (4), (5) and (6), on the one hand, and their holomorphic transforms (9), (10) and (11), on the other. Under the transformation (8), the vacuum state $|0\rangle$ is mapped into itself, and the coherent states $|z^l\rangle$ are mapped into the coherent states $|z^l\rangle$. Therefore the notion of coherence is global for all observers on \mathcal{C} , *i.e.*, any two observers will agree on what is a coherent state *vs.* what is a noncoherent state. A consequence of this fact is the following. Under holomorphic diffeomorphisms of \mathcal{C} , the semiclassical regime of the quantum theory on \mathcal{H} is mapped into the semiclassical regime, and the strong quantum regime is mapped into the strong quantum regime.

3 Local coherent states

We now relax the conditions imposed on \mathcal{C} . In this section we will assume that \mathcal{C} carries an almost complex structure $J_{\mathcal{C}}$ compatible with the symplectic structure $\omega_{\mathcal{C}}$. Specifically, an almost complex structure is defined as a tensor field $J_{\mathcal{C}}$ of type $(1,1)$ such that, at every point of \mathcal{C} , $J_{\mathcal{C}}^2 = -\mathbf{1}$. Using Darboux coordinates q^l, p_l on \mathcal{C} let us form the combinations

$$w^l = q^l + ip_l, \quad l = 1, \dots, n. \quad (12)$$

Compatibility between $\omega_{\mathcal{C}}$ and $J_{\mathcal{C}}$ means that we can take $J_{\mathcal{C}}$ to be

$$J_{\mathcal{C}} \left(\frac{\partial}{\partial w^l} \right) = i \frac{\partial}{\partial w^l}, \quad J_{\mathcal{C}} \left(\frac{\partial}{\partial \bar{w}^l} \right) = -i \frac{\partial}{\partial \bar{w}^l}. \quad (13)$$

Unless \mathcal{C} is a complex manifold to begin with, equations (12) and (13) fall short of defining a complex structure $\mathcal{J}_{\mathcal{C}}$. The set of all such w^l does not provide a holomorphic atlas for \mathcal{C} . There exists at least one canonical coordinate transformation between Darboux coordinates, call them (q^l, p_l) and (q'^l, p'_l) , such that the passage between $w^l = q^l + ip_l$ and $w'^l = q'^l + ip'_l$ is given by a nonholomorphic function g in n variables,

$$w' = g(w, \bar{w}), \quad \bar{\partial}g \neq 0. \quad (14)$$

Mathematically, nonholomorphicity implies the mixing of w^l and \bar{w}^l . Quantum-mechanically, the loss of holomorphicity has deep physical consequences. One would write, in the initial coordinates w^l , a defining equation for the vacuum state $|0\rangle$

$$a^l|0\rangle = 0, \quad l = 1, \dots, n, \quad (15)$$

where $a^l = Q^l + iP_l$ is the corresponding local annihilation operator. However, one is just as well entitled to use the new coordinates w'^l and write

$$a'^l|0'\rangle = 0, \quad l = 1, \dots, n, \quad (16)$$

where we have primed the new vacuum, $|0'\rangle$. Are we allowed to identify the states $|0\rangle$ and $|0'\rangle$? We could identify them if w^l were a holomorphic function of w^l ; such was the case in section 2. However, now we are considering a nonholomorphic transformation, and we cannot remove the prime from the state $|0'\rangle$. This is readily seen. We have

$$a' = G(a, a^\dagger), \quad (17)$$

with G a quantum nonholomorphic function corresponding to the classical nonholomorphic function g of equation (14). As $[a^j, a_k^\dagger] = \delta_k^j$, ordering ambiguities will arise in the construction of G from g , that are usually dealt with by normal ordering. Normal ordering would appear to allow us to identify the states $|0\rangle$ and $|0'\rangle$. However this is not the case, as there are choices of g that are left invariant under normal ordering, such as the sum of a holomorphic function plus an antiholomorphic function, $g(w, \bar{w}) = g_1(w) + g_2(\bar{w})$. Under such a transformation one can see that the state $|0\rangle$ satisfying eqn. (15) will not satisfy eqn. (16). We conclude that, in the absence of a complex structure on classical phase space, the vacuum depends on the observer. The state $|0\rangle$ is only defined locally on \mathcal{C} ; it cannot be extended globally to all of \mathcal{C} .

Similar conclusions may be expected for the coherent states $|w^l\rangle$. The latter are defined only locally, as eigenvectors of the local annihilation operator, with eigenvalues given in equation (12):

$$a^l |w^l\rangle = w^l |w^l\rangle, \quad l = 1, \dots, n. \quad (18)$$

Due to $[a^j, a_k^\dagger] = \delta_k^j$, under the nonholomorphic coordinate transformation (14), the local coherent states $|w^l\rangle$ are *not* mapped into the local coherent states satisfying

$$a'^l |w'^l\rangle = w'^l |w'^l\rangle, \quad l = 1, \dots, n \quad (19)$$

in the primed coordinates. No such problems arose for the holomorphic operator equation $A' = F(A)$ corresponding to the holomorphic coordinate change $z' = f(z)$ of equation (8), because the commutator $[A^j, A_k^\dagger] = \delta_k^j$ played no role. Thus coherence becomes a local property on classical phase space. In particular, observers on \mathcal{C} not connected by means of a holomorphic change of coordinates need not, and in general will not, agree on what is a semiclassical effect *vs.* what is a strong quantum effect.

4 Coherence

We have called *coherent* the states constructed in previous sections. However, we have not verified that they actually satisfy the usual requirements imposed on coherent states [3]. What ensures that the states so constructed are actually coherent is the following argument.

We have made no reference to coupling constants or potentials, with the understanding that the Hamilton–Jacobi method has already placed us, by means

of suitable coordinate transformations, in a coordinate system on \mathcal{C} where all interactions vanish [4]. Then any dynamical system with n independent degrees of freedom that can be transformed into the freely-evolving system can be further mapped into the n -dimensional harmonic oscillator. The combined transformation is canonical. Moreover it is locally holomorphic when \mathcal{C} is an almost complex manifold. Thus locally on \mathcal{C} , our global states $|z^l\rangle$ of section 2 coincide with the coherent states of the n -dimensional harmonic oscillator. Mathematically this fact reflects the structure of a complex manifold: locally it is always holomorphically diffeomorphic to \mathbf{C}^n [4]. Physically this fact reflects the decomposition into the creation and annihilation modes of perturbative quantum mechanics and field theory. In this way, the mathematical problem of patching together different local coordinates z_α^l labelled by an index α may be recast in physical terms. It is the patching together of different local expansions into creators A_α^\dagger and annihilators A_α , for different values of α .

In particular, we can write the resolution of unity on \mathcal{H} associated with a holomorphic atlas on \mathcal{C} consisting of coordinates z_α^l :

$$\sum_{\alpha} \sum_{l=1}^n \int_{\mathcal{C}} d\mu_{\mathcal{C}} |z_{\alpha}^l\rangle \langle z_{\alpha}^l| = \mathbf{1}, \quad (20)$$

where $d\mu_{\mathcal{C}}$ is an appropriate measure on \mathcal{C} .

Analogous arguments are also applicable to the local states $|w^l\rangle$ of section 3. In particular, every almost complex manifold is locally a complex manifold. Every holomorphic coordinate chart on \mathcal{C} is diffeomorphic to \mathbf{C}^n , so the $|w^l\rangle$ look locally like the coherent states of the n -dimensional harmonic oscillator. However, the loss of holomorphicity of \mathcal{C} alters equation (20) in one important way. We may write as above

$$\sum_{\alpha} \sum_{l=1}^n \int_{\mathcal{C}} d\mu_{\mathcal{C}} |w_{\alpha}^l\rangle \langle w_{\alpha}^l|, \quad (21)$$

but we can no longer equate this to the identity on \mathcal{H} . The latter is a *complex* vector space, while eqn. (21) allows one at most to expand an arbitrary, real-analytic function on \mathcal{C} , since the latter is just a real-analytic manifold. Hence we cannot equate (21) to $\mathbf{1}_{\mathcal{H}}$. We can only equate it to the identity on the *real* Hilbert space of real-analytic functions on \mathcal{C} . This situation is not new; coherent states without a resolution of unity have been analysed in ref. [5], where they have been related to the choice of an inadmissible fiducial vector.

5 Discussion

The purpose of this article is to present a framework for quantum mechanics in which coherent states are defined locally on classical phase space \mathcal{C} , but not necessarily globally. This is an explicit implementation of the relativity (underlying the notion of duality) of the concept of a quantum.

Our results may be summarised as follows. Coherent states can always be defined locally, *i.e.*, in the neighbourhood of any point on \mathcal{C} . When there is a complex structure $\mathcal{J}_{\mathcal{C}}$, coherence becomes a global property on \mathcal{C} . In the absence of a complex structure, however, the best we can do is to combine Darboux coordinates q, p as $q + ip$. Technically this only defines an almost complex structure $J_{\mathcal{C}}$ on \mathcal{C} . Since the combination $q + ip$ falls short of defining a complex structure, quantities depending on $q + ip$ on a certain coordinate patch will generally also depend on $q - ip$ when transformed to another coordinate patch. This proves that coherence remains a local property on classical phase space: observers not connected by means of a holomorphic change of coordinates need not, and in general will not, agree on what is a semiclassical effect *vs.* what is a strong quantum effect.

Most physical systems admit a complex structure $\mathcal{J}_{\mathcal{C}}$ on their classical phase spaces \mathcal{C} . Prominent among them is the 1-dimensional harmonic oscillator. Mathematically, the corresponding \mathcal{C} supports the simplest holomorphic structure, that of the complex plane. Physically, canonical quantisation rests on the decomposition of a field into an infinite number of oscillators. The notions that the vacuum state is unique, and that coherence is a universal property independent of the observer, follow naturally. However, recent breakthroughs in quantum field theory and M-theory suggest the need for a framework in which duality transformations can be accommodated at the elementary level of quantum mechanics, before considering field theory or strings. This, in turn, would help to understand better the dualities underlying quantum fields, strings and branes.

The formalism presented here can accommodate duality transformations in a natural way. In the absence of a complex structure $\mathcal{J}_{\mathcal{C}}$, all our statements concerning the vacuum state and the property of coherence are necessarily local in nature, *i.e.*, they do not hold globally on \mathcal{C} . A duality transformation of the quantum theory on \mathcal{H} will thus be specified by a nonholomorphic coordinate transformation on \mathcal{C} .

However, the question immediately arises: do we not have an overabundance of vacua? Does every imaginable nonholomorphic transformation induce a *physical* duality? Duality is not to be understood as a transformation between different physical phenomena. Rather, it is to be understood as a transformation between different descriptions of the same quantum physics. A judicious application of physical symmetries can vastly restrict this apparent overabundance of vacua. Usually dualities appear under the form of a group \mathcal{D} . Rather than taking every imaginable nonholomorphic transformation to define a physical duality we must assume, as is the case in M-theory, a knowledge of the duality group \mathcal{D} , and restrict ourselves to those nonholomorphic transformations that actually realise it.

In using coherent states our analysis has been geometric. Geometric approaches to quantum mechanics [6, 7, 8, 9, 10] have proved extremely useful. In particular, that coherence equals holomorphicity has been known for long [11]. Here we have proved that noncomplex structures (such as almost complex structures) allow to implement duality transformations.

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